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ALL TRIANGULATIONS OF THE PROJECTIVE PLANE ARE GEOMETRICALLY REALIZABLE IN E^4

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ABSTRACT

We say that a triangulation T of a manifold is geometrically realizable in E^n provided there is a triangulation $T' \subset E^n$ isomorphic to T such that each simplex of T' is convex. A triangulation of the projective plane cannot be geometrically realizable in E^3 because it is not topologically realizable in E^3. We show that it is, however, geometrically realizable in E^4.

1. Introduction

A *convex* 3*-dimensional polytope* (hereafter to be called a 3*-polytope*) is a 3-dimensional set that is the convex hull of a finite set of points. The *graph* of a 3-polytope P is defined to be the graph consisting of the vertices and edges of P. Graphs of other structures that we shall be dealing with, such as triangulations, are defined in the same way.

A theorem of Steinitz [3, 2, ch. 13] characterizes the graphs of 3-polytopes as those that are planar and 3-connected. It follows from Steinitz's theorem that given any triangulation of the 2-sphere there exists a 3-polytope isomorphic to it. Another way of looking at this is that any triangulation of the 2-sphere can be realized with convex triangles in E^3. For triangulations of other orientable 2-manifolds it is not known whether all of them can be realized with convex triangles in E^3. None of the nonorientable triangulations can be realized in E^3, while every simplicial 2-dimensional complex can be realized in E^5 as a subcomplex of the boundary of a suitably closed 5-dimensional convex polytope (see [2, ch. 7]). In this paper we prove that the triangulations of the projective plane can be realized with convex triangles in E^4.

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2. Graphs and complexes

By a *path* in a graph we mean a simple arc consisting of edges of the graph. The vertices at the two ends of the arc are called its *endpoints*. Any vertex other than an endpoint is called an *interior vertex*. The *length* of a path is the number of edges in it. A *circuit* is a simple closed curve consisting of edges of the graph. The *length* of a circuit is the number of edges in it. A circuit of length n will be called an n*-circuit*. If an edge has vertices x and y, we denote it xy. If a circuit has edges $ab, bc, cd, \dots$ we denote it by $abcd\dots$.

A *cell complex* is a collection $\mathscr{C}$ of convex polygons such that the intersection of any two polygons is empty, a vertex of both or an edge of both. If all polygons are triangles we call it a *simplicial cell complex*. This definition differs slightly from the usual definition of cell complex, but since we are dealing only with 2-dimensional complexes our definition is a little less cumbersome. When a collection of polygons intersects in this way we say that they *meet properly*. The polygons, their edges and their vertices will be called the *faces* of the complex. For $0 \leq n \leq 2$, we use the term n*-face* to mean n-dimensional face. The term *facet* will be used for 2-faces of 3-polytopes. Two cell complexes are *isomorphic* provided there is a one-to-one, incidence preserving, dimension preserving function taking the set of faces of one complex onto the set of faces of the other.

A *topological cell complex* is a collection of polygons with various collections of edges and vertices identified such that:

(i) An edge of a polygon is identified with at most one edge of any other polygon, and an edge of a polygon P is not identified with any other edge of P.

(ii) A vertex of a polygon is identified with at most one vertex of any other polygon, and a vertex of a polygon P is not identified with any other vertex of P.

(iii) If an edge e is identified with an edge e', then the vertices of e are identified with the vertices of e'.

(iv) If two vertices of a polygon P are identified with two vertices of polygon P' then the two vertices are joined by an edge of each polygon and these edges are identified.

A topological cell complex $\mathscr{C}$ is *isomorphic* to a cell complex $\mathscr{C}'$ provided there is a one-to-one correspondence between the polygons, such that two polygons in $\mathscr{C}$ intersect on an edge or vertex if and only if the two corresponding polygons in $\mathscr{C}'$ have identified edges or vertices respectively. In this case we say that $\mathscr{C}'$ is a *geometric realization* of $\mathscr{C}$.

A set S obtained from a topological cell complex by the identification topology, and any set homeomorphic to S will also be called a *topological cell* *complex*. The *faces* are the images of the vertices, edges and polygons under the homeomorphism.

Hereafter we shall use the term 2-*complex* (sometimes just "complex") for 2-dimensional topological cell complex.

An edge of a 2-complex is a *boundary edge* provided it belongs to exactly one polygon in the complex. The *boundary* of a 2-complex is the union of its boundary edges. An edge of a 2-complex that is not a boundary edge but joins two vertices of the boundary is called a *diagonal* of the complex. If a 2-complex $\mathscr{C}$ is a subcomplex of a 2-complex $\mathscr{C}'$, then any edge in $\mathscr{C}'$ that is not an edge of $\mathscr{C}$ but joins two boundary vertices of $\mathscr{C}$ is called an *outer diagonal* of $\mathscr{C}$. Note that an outer diagonal of a complex is not a diagonal of the complex.

A triangulated disc is a type of 2-complex that we shall be dealing with. One can add the disc to the complex (with the obvious identifications) to get a new complex that is a topological sphere. This can be done provided the triangulated disc has no diagonals. If there were such a diagonal then condition (iv) would not be met and we would not have a complex. The following theorem of Steinitz (see [3]) tells us more about such a spherical complex:

THEOREM (Steinitz). *Any* 2-*complex homeomorphic to a sphere is isomorphic to the cell complex consisting of the boundary of a* 3-*polytope.*

There are other complexes that are topological discs besides those that are triangulations. Any such disc such that adding the disc to the complex with the obvious identifications creates a spherical complex, will be called a *Schlegel complex*. Any 3-polytope P whose boundary complex is isomorphic to this sphere will be said to be *isomorphic* to the Schlegel complex. Thus any triangulated disc without diagonals is a Schlegel complex. The disc that is added to the complex will be called the *outer face* of the complex. In a 3-polytope isomorphic to a Schlegel complex $\mathscr{C}$, the facet of the polytope corresponding to the outer face of $\mathscr{C}$ will be called the *outer facet* of P. By a theorem of Barnette and Grünbaum [1] there is a polytope isomorphic to P with the outer facet congruent to any preassigned polygon with the same number of edges.

The boundary of P minus its outer facet will be called a *polyhedral realization* of $\mathscr{C}$.

For a vertex v of a 2-complex $\mathscr{C}$ we define the *star* of v in $\mathscr{C}$ to be the union of the faces of $\mathscr{C}$ that meet v. The *antistar* of v in $\mathscr{C}$ is the union of the faces of $\mathscr{C}$ not meeting v. The *link* of v in $\mathscr{C}$ is the intersection of the star and the antistar.

Let v be a vertex of a simplicial 2-complex $\mathscr{C}$ such that the star of v is homomorphic to a disc. We consider two sets of vertices A and B in the link of v

such that $A \cap B$ consists of 2 vertices, if star v is a disc, and at most two vertices otherwise, and such that the vertices in both A and B are consecutive in their cyclic ordering around v. We replace the star of v by two points v' and v'', together with triangles that are the convex hulls of v' and edges joining vertices of A, or the convex hulls of v'' and the edges joining vertices of B, or the convex hulls of the segment $v'v''$ with vertices in $A \cap B$. This process, called *vertex splitting*, replaces a cell in $\mathscr{C}$ by a cell and thus does not alter the topological type of the complex. The reader may check that this process does always produce a complex. It is not true that this process will always produce a cell complex. However, if the new vertices v' and v'' are sufficiently close to v, and if $\mathscr{C}$ is the boundary of a 3-polytope, then this vertex splitting process produces another cell complex that is the boundary of a 3-polytope.

When a splitting is done on a cell complex to produce another cell complex we call it a *geometric vertex splitting*.

The following well known lemma does not appear to be in the literature.

LEMMA 2. *Any vertex splitting on the boundary of a simplicial 3-polytope P can be done geometrically.*

PROOF. Let v be the vertex to be split and let the sets A and B be as in the definition of vertex splitting. Let H be a plane that separates v from the other vertices of P. The intersection of H with P is a convex polygon Q whose edges project onto the edges of the link of v. Let the common vertices of A and B be x and y. Let H' be the plane determined by v, x and y. Since $H \cap P$ is a convex polygon, H' will intersect its boundary at just the points corresponding to x and y. Thus the boundary of $H \cap P$ is divided into two paths one on each side of H'. The vertices of these paths project onto A and B, thus each of the sets A and B lie on just one side of H'. To accomplish the splitting, we choose v' to be the vertex v. The new facets meeting v' will lie on just one side of H'. We choose v'' to be a point close to v but on the other side of H'. Thus the new facets meeting v', and the new facets meeting the edge $v'v''$ will lie on the opposite side of the plane H', and we have the new facets meeting properly.

It is important to notice two things about the construction in our proof. First, by choosing v' close enough to v we may construct our new polytope within any prescribed ε of the original (in the Hausdorff metric). Second, the facets of the new polytope that do not meet v'' are facets of the original polytope. This will be very important in later constructions.

The inverse operation to vertex splitting will be called *edge shrinking*. It is not true that any edge can be shrunk. In a simplicial 2-complex all of the conditions of a complex are clearly met by the collection of triangles produced by the shrinking except possibly condition (iv). Condition (iv) would fail to be met if in the original complex an edge e meets v', an edge e' meets v'', and e and e' meet at some other vertex, and are not the edges of a face containing the edge $v'v''$. This is clearly the only condition that would prevent the formation of a complex by edge shrinking, thus we have:

LEMMA 3. *An edge $v'v''$ of a simplicial 2-complex $\mathscr{C}$ is shrinkable provided the edge $v'v''$ does not belong to a 3-circuit that does not bound a face of $\mathscr{C}$.*

Another way of looking at this is that shrinking an edge of a triangulation produces a cell complex provided a double edge is not created. If shrinking an edge produces another complex we say that the edge is *shrinkable*. A 3-circuit that does not bound a face will be called a *nonfacial triangle*.

3. Polygon realizations of Möbius strips

In a triangulated Möbius strip there will be two essentially different kinds of diagonals. A diagonal may have the property that if the strip is cut along this diagonal the result is that the strip has been changed into a cell. Such an edge we shall call a *cut edge* and we shall reserve the term *diagonal* for those that have the property that cutting along them produces two pieces, a cell and a Möbius strip. If we have a triangulation in which every edge not on the boundary is a cut edge we shall call it a *simple triangulation*. A simple triangulation in which every vertex has valence four will be called an elementary triangulation. The reader may easily verify the following:

LEMMA 4. *Any elementary triangulation of the Möbius strip has an odd number of vertices.*

We shall be dealing with a special kind of geometric realization of triangulated Möbius strips:

A geometric realization R of a triangulated Möbius strip M is a *polygon realization* provided the orthogonal projection of R onto some plane is a polygon P such that the boundary of R is taken one-to-one onto the boundary of P.

In triangulations of the projective plane the presence of one particular type of 4-circuit will require special arguments. A 4-circuit in a triangulated projective plane bounding a cell C, where C is not the union of two triangles, is called a *special 4-circuit*, provided it has two outer diagonals in the triangulation.

THEOREM 1. *The elementary triangulation T_{2n+1} of the Möbius strip with $2n+1$ vertices has a polygon realization in E^4.*

PROOF. Let P be a $(2n+1)$-gon in the xy-plane in E^4. Let the vertices of P be labeled $a_1, a_3, a_5, \cdots, a_{2n+1}, a_2, a_4, \cdots, a_{2n}$ in a cyclic ordering. For each i, $1 < i < 2n$, we choose b_i to be the point in E^4 with the first two coordinates the same as a_i, the third coordinate equal to i and the fourth coordinate 0. Any two consecutive triangles in this set lie on different planes so they meet properly. For any two nonconsecutive triangles the one with a vertex whose third coordinate is largest has all of its relative interior points above all relative interior points of the other, thus they also meet properly. It follows that the triangles $b_ib_{i+1}b_{i+2}$, for $1 < i < 2n-2$, form a triangulated strip that is isomorphic to the complex consisting of the first $2n-2$ triangles of T_{2n+1}.

We complete the polygon realization of T_{2n+1} by choosing b_{2n+1} to have its first two coordinates the same as a_{2n+1}, its third coordinate arbitrary, and its fourth coordinate different from 0. We add the triangles $b_{2n-1}b_{2n}b_{2n+1}$, $b_{2n}b_{2n+1}b_1$ and $b_{2n+1}b_1b_2$. None of these triangles will meet any previous ones improperly because their relative interiors are not in the 3-space spanned by the previous triangles. The edges of the polygon correspond to the edges of the boundary of this Möbius strip by the obvious correspondence of subscripts, so the boundary of the Möbius strip will project onto the edges of the polygon. Since no two edges of the strip correspond to the same edge of the polygon the projection of the boundary of the strip is one-to-one.

THEOREM 2. *Any simple triangulation T of a Möbius strip has a polygon realization.*

PROOF. If the triangulation is elementary we are done by the previous theorem. Suppose that the triangulation is not elementary. Our proof is inductive. The induction is started by the elementary triangulations and by the special nonelementary triangulation in Fig. 1. The induction is on the number of vertices of T.

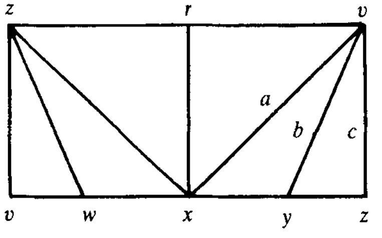

Fig. 1.

(We shall represent triangulations of the Möbius strip as they appear when cut along a cut edge, thus our figures are triangulated quadrilaterals with opposite edges identified.)

Let v be a vertex of T of valence greater than 4.

Case I. The valence of v is at least 6. In this case let the first four edges meeting v, counting in a cyclic ordering starting at an edge on the boundary of the strip, be va, vb, vc and vd (Fig. 2). The edges vb, vc and vd are cut edges. We shall shrink the edge cd to produce a smaller triangulation. To see that we get a triangulation after the shrinking, note that there are cut edges on each side of this edge emanating from v. This prevents the edge ad from belonging to a nonfacial triangle.

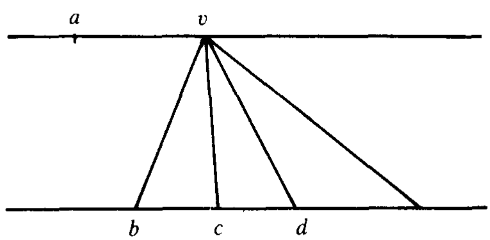

Fig. 2.

To get a realization of our original triangulation, suppose that the new triangulation T' resulting from the edge shrinking is realizable by a cell complex R projecting onto a polygon P. We can easily get a realization of T by choosing a triangle F bounded by two cut edges meeting the vertex of R corresponding to v, and an edge e of the boundary. We then choose a point p very near the midpoint of e. We choose it near enough that introducing p as a new vertex and replacing F by two triangles (with a common edge pv) will produce a cell complex R'.

Let e' be the edge of P that e projects onto. We may also choose p so that it projects onto a point p' near the midpoint of e' but not in P. By choosing the point close enough to e the projection of p will be a vertex of con$\{p', P\}$, thus we have a polygon realization of T.

Case II. v has valence 5, and the triangulation is not the triangulation in Fig. 1.

Let the cut edges meeting v be a, b and c, with endpoints x, y and z as in Fig. 3. We wish to shrink edge xy and proceed as in the previous case. It may happen, however, that an edge of T joins x and z, preventing us from shrinking edge xy. Such an edge cannot be an edge of the boundary of the strip because as we

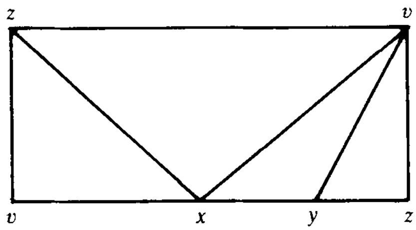

Fig. 3.

traverse the boundary from x to z we pass the vertex v. Thus the edge xz is a cut edge. The vertices x, z and v cannot determine a triangle of the triangulation because xv would then be a double edge (see Fig. 3).

Since there can be no interior vertices in the strip, there must be an edge zw with w between x and v as in Fig. 1. There can be only one vertex on this path from x to v, for if there were another such vertex it would be joined to z, and we would have Case I. By a similar argument, there is one vertex r, between z and v as shown, with r joined to x. We now have a complete description of the triangulation, and it is the triangulation in Fig. 1.

Case III. T is the triangulation in Fig. 1. In this case we construct a polygon realization.

Let P be a polygon in the xy-plane onto which we are to project the Möbius strip. We label the vertices of P $a_1, \cdots, a_6$ in cyclic order. We choose b_6 to have the same first two coordinates as a_6, third coordinate different from 0, and fourth coordinate equal to 0. Let E be the 3-space spanned by the xy-plane and b_6. In E we take the triangles $a_2a_3a_5$ and $a_3a_5b_6$ (these are to correspond to the triangles rvx and vxy). We choose b_1 to have the same first two coordinates as a_1, second coordinate arbitrary and third coordinate different from 0. We now add the triangles $b_1a_2a_5$, $b_1a_4a_5$, $b_1a_3a_4$ and $b_1a_3b_6$ (these correspond to the triangles meeting z). The result is a triangulated Möbius strip isomorphic to T with the isomorphism taking z to b_1, r to a_2, v to a_3, w to a_4, x to a_5 and y to b_6.

4. Realizations of the projective plane

Parts of the realizations that we construct will come from the boundary complexes of 3-polytopes. The following lemmas although not new have probably not appeared in these particular forms.

LEMMA 5. *If G is the graph of a simplicial 3-polytope that is not the graph of a tetrahedron, then every vertex meets a shrinkable edge.*

PROOF. If we have a 3-valent vertex then it is obvious that every edge meeting it is shrinkable. Suppose that v is a vertex of valence more than three. An edge meeting v will fail to be shrinkable only if it belongs to a nonfacial triangle. One edge of such a triangle would be an outer diagonal of the star of v. Among all such outer diagonals we choose an edge e such that a path P along the link of v joining the endpoints of e has minimal length. The path P must have an interior vertex or else the graph would have a double edge. Let x be an interior vertex of this path. The edge xv cannot belong to a nonfacial triangle because one edge of such a triangle would have to cross e. Thus xv is a shrinkable edge.

LEMMA 6. *Let F and G be two facets of a simplicial 3-polytope P, meeting on an edge. Given any $\varepsilon < 0$ and any 3-simplex S with facets F' and G' there is a 3-polytope P' isomorphic to P, within ε of S (in the Hausdorff metric), containing F' and G' as the facets corresponding to F and G under the isomorphism.*

PROOF. We begin by shrinking edges in the graph of P. If there are no vertices of P missing F and G, then P is a simplex and the conclusion is obvious. If there is a vertex of P missing these two facets then there is a shrinkable edge meeting such a vertex. We choose such an edge and shrink it. Repeatedly doing these shrinkings reduces the graph of P to the graph of the tetrahedron. We now take the tetrahedron S and do the corresponding geometric vertex splittings to produce a polytope isomorphic to P.

Since all of these splittings can be done so that the new vertices are not vertices of the facets corresponding to F or G, we can do all of the splittings so that F' and G' are not changed.

THEOREM 3. *Any triangulation T of the projective plane can be geometrically realized in E^4.*

PROOF. We first make two reductions in the problem.

Reduction I. It is sufficient to consider triangulations in which every circuit of length three which bounds a cell bounds a triangle of the triangulation. Consider any triangulation with circuits of length three bounding cells that are not triangles of the triangulation. Replacing the cells bounded by these circuits by triangles produces another triangulation M'. Let A be a triangle in a geometric realization of T, corresponding to such a 3-circuit. Any such cell in the original triangulation bounded by such a circuit C is a Schlegel complex isomorphic to a simplicial 3-polytope P (by Steinitz's Theorem). By Lemma 6 there exists a 3-polytope P' isomorphic to P with C bounding one facet F and with P' within any preassigned ε of the triangle spanned by C. We choose ε small enough so that replacing the triangle A by the boundary of P' minus A guarantees that faces meet properly, thus creating a cell complex. Doing this for each such circuit of length three yields the desired realization.

Reduction II. It is sufficient to consider triangulations in which the only nonspecial 4-circuits bounding cells are those bounding cells consisting of two triangles meeting on an edge.

Suppose that a nonspecial 4-circuit $C = abcd$ bounds a cell A' that is not one consisting of two triangles meeting on an edge and lacks an outer diagonal, say, ac. To the complex A' we add the triangles adb and cdb. We now have a 2-sphere isomorphic to a 3-polytope, by Steinitz's Theorem.

In the original triangulation we shall remove A' and replace it with the triangles acb and acd. This produces a complex because ac was not an edge before this replacement. We shall call this triangulation T'.

Suppose we have a geometric realization R of T'. For any vertex x of T' we shall let x' be the corresponding vertex of R.

The convex hull of a', b', c' and d' is a simplex S. By Lemma 6 we can construct a 3-polytope P isomorphic to S within ε of S, with $a'd'b'$ and $b'd'c'$ as facets corresponding to adb and bdc in X. We remove the facets $a'b'd'$ and $c'b'd'$ from the boundary of P. If we have chosen ε small enough then the facets remaining in the boundary of P will meet the other triangles of T' properly because they will lie within ε of faces acb and acd of T', and we have the desired realization.

We shall call 3-circuits that bound cells that are not triangles, and nonspecial 4-circuits that bound cells that are not two triangles meeting on an edge, *forbidden circuits*. We now shall consider triangulations without forbidden circuits.

For these triangulations we show that they can be decomposed into two complexes — one that is a Schlegel complex that contains all of the vertices of the triangulation, and the other complex a Möbius strip with a polygon realization. Then we shall show how to glue geometric realizations of them together to get a realization of T.

To show that the above decomposition of T exists it suffices to show that there exists a Möbius strip in T with a simple triangulation with a polygon realization and that does not admit any outer diagonals. If such a Möbius strip exists, then the complementary complex will be a triangulated cell without diagonals and will thus be a Schlegel complex.

The first case that we treat is triangulations that have a special 4-circuit bounding a cell C. Let the vertices of the 4-circuit in cyclic order around C be a, b, c and d. The cell C together with the cell bounded by the quadrilateral $adbc$ forms a Möbius strip in the projective plane. The Möbius strip has no outer diagonals because each two of the four vertices on the boundary of the strip are joined by edges of the strip.

The link of a in this strip consists of two paths — a path A from c to d inside $adbc$, and a path B from d to b inside $abcd$ (see Fig. 4). We take the cell C_1 bounded by A, db and bc and construct a sphere S from it by adding a face F, bounded by A and a new edge cd, and also adding a triangular face cdb. The sphere now consists of a nontriangular face F and a complementary triangulated cell. This triangulated cell has no diagonals because such a diagonal would be an edge connecting two nonconsecutive vertices of A. There cannot be such an edge because in the original triangulation, this edge together with two edges meeting a would form a forbidden circuit.

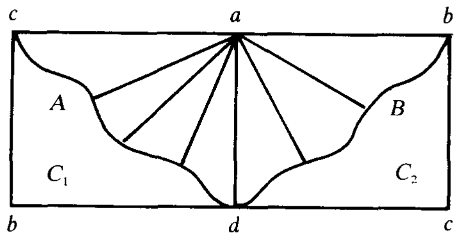

Fig. 4.

It follows that the sphere S is 3-polyhedral. We do a similar construction with the cell C_2 bounded by B, bc and cd, obtaining a 3-polyhedral sphere S'. We now choose two 3-polytopes P and P', isomorphic to S and S' respectively, and using a projective transformation if necessary, situate them so that their boundaries intersect on the triangular face cbd, and such that a projection in some direction will project the triangle cbd onto the boundary of the projected image (we may in fact choose the images of the projection to be three vertices of some specified quadrilateral Q). Next we choose a point a' in E^4, but not in the 3-space spanned by P and P', which projects onto the fourth vertex of Q.

Our polygon realization of the Möbius strip is the union of the cells on the boundaries of the 3-polytopes of P and P' corresponding to C_1 and C_2 together with the convex hulls of a' and the edges of the paths corresponding to A and B. We have established that this Möbius strip has a polygon realization and thus in the case where special 4-circuits exist we have the desired decomposition.

We now assume that T has no special 4-circuits. We begin by taking the antistar of a vertex in T. This set is a Möbius strip without outer diagonals. We shall show how to repeatedly discard parts of this strip until we are left with one that has a simple triangulation and no outer diagonals.

If at any step of our construction we have a strip with a diagonal, then that diagonal together with part of the boundary of the strip will bound a cell. This cell we can discard and preserve the property of having no outer diagonals. We shall call this *discarding a diagonal.*

If at some stage we have discarded all diagonals and we have a vertex on the boundary of the strip such that no cut edges meet that vertex, then we may remove the triangles of the star of the vertex in the strip and obtain a smaller Möbius strip without outer diagonals. We call this *discarding a star.*

Suppose, now, that we have a strip for which we cannot discard a diagonal or star. Suppose also that there is an interior vertex (i.e. one not on the boundary of the strip). Since the graph of a triangulated Möbius strip is connected, there exists a vertex v on the boundary joined to an interior vertex x. Since no star is removable, there is a cut edge vy meeting v. We treat two cases.

Case I. There is another cut edge vw such that vx is between vy and vw in the strip (see Fig. 5). Let P be the path along the boundary of the strip from y to w that misses v. Assume that we have chosen the vertices y and w such that the length of P is minimal. The path P cannot be an edge because then vwy would be a forbidden 3-circuit. If, however, there is an interior vertex p of P then by the minimality condition on y and w, there would be no cut edge from p to v. This means that there is no cut edge from p to any vertex, because edges from p to other vertices would be diagonals. It follows that the star of p is removable.

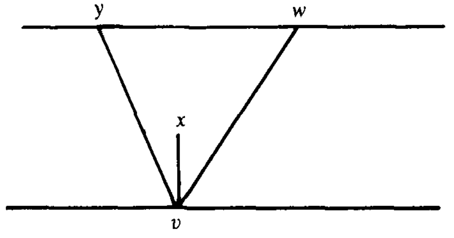

Fig. 5.

Case II. There is no cut edge as described in Case I. Let vs be an edge on the boundary of the strip such that vx is between vs and vy in the strip (see Fig. 6). Since stars are not removable, there is a cut edge st meeting s. Let P' be the path along the boundary of the strip from y to t missing v. Assume that y and t

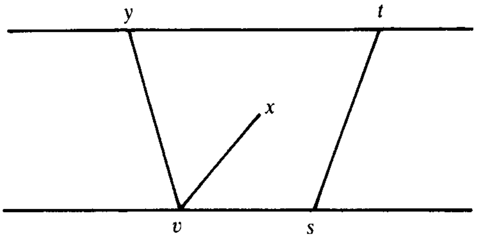

Fig. 6.

are chosen such that P' has minimal length. The path P' cannot be an edge because then *vyts* would be a forbidden 4-circuit. On the other hand, if there is an interior vertex p'' on P then by the minimality condition on y and t, p'' would have no cut edges meeting it and thus its star would be removable.

Thus, repeated applications of diagonal and star removal results in a triangulated Möbius strip with no diagonals, no interior vertices and no outer diagonals, that is, a Möbius strip with a simple triangulation and no outer diagonals.

Let T be any triangulation of the projective plane without forbidden circuits and let it be decomposed into a Möbius strip S and Schlegel complex C. Let P be a 3-polytope isomorphic to C with outer face F. By Theorem 2 there is a polygon realization of S in E^4 that projects onto an n-gon Q. We can choose a 3-polytope P' isomorphic to P with the outer face F congruent to Q. By a linear transformation we can obtain a realization R of S that is within a given ε of F.

Each triangle in R projects onto a triangle (or segment) in F. Since F meets other facets of P' properly, these images meet the other facets of P' properly. Thus if the vertices of these images are moved a small enough distance the resulting triangles will still meet the other facets of F properly (note that some of the vertices moved may be vertices of F). In other words, by choosing R close enough to F we may move the vertices of F to their corresponding vertices in R and have faces meet properly. Moving these vertices of F produces a gluing of R to P' and we have our realization of T.

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